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Letter to the Editor

A compact formulation for conditioned spectral density function analysis by means of the $\mathbf{LDL}^H$ matrix factorization

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1. Introduction

Several methods have been developed in the last decades to deal with the subject of transmission path analysis (TPA) in noise and vibration problems. A distinction can be made between the so-called one-step methods and two-step methods. The MISO method [1] is a one-step TPA method because it only requires operational measurements among subsystems in a linear N-dimensional network. That is to say, the method allows factorization of the signal (usually acceleration, velocity or displacement in a given direction, or the acoustic pressure at a given location) at one network subsystem in terms of the signals or forces at the remaining ones, with the use of only operational measured data. This is to be compared with two-step TPA methods like the GTDT method (global transfer direct transfer, see Ref. [2]) or the FTF method (force transfer functions, see Refs. [2–4]), which require transfer functions to be measured initially with the network stationary. Operational measurements are carried out in a second step and the previously measured transfer functions are then used to obtain the desired signal factorizations.

The bases of most TPA methods were developed in the mid 1970s. Since then much work has been done in order to solve some of their numerical problems, as well as to enlarge their range of applicability (refer to Refs. [5–8] concerning the FTF methods and Refs. [9–11] concerning the GTDT methods). In this paper, attention will be paid to the MISO method. It will be shown that the conditioned spectral density function analysis developed to deal with partially correlated signals on a linear network corresponds in fact, to the $\mathbf{LDL}^H$-factorization of the network signal cross-spectra matrix $\bar{\tilde{\mathbf{S}}}$. Although this may be a recognized result because the MISO method dates from the 1970s (see Ref. [1]) the authors have not found any published proof of it. A proof is derived in this paper that might be found interesting by itself and serve as a compendium to obtain the MISO factorizations in a compact and straightforward way.

Concerning the notation used in this paper, the term subsystem has been identified with a single-degree-of-freedom (d.o.f.) of a physical entity and not with the physical entity itself. This

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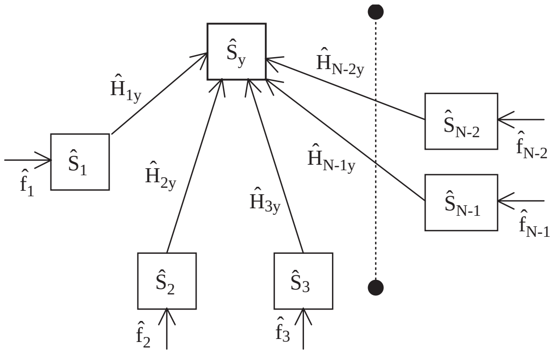

Fig. 1. The MISO network. First factorization: signal or force frequency response functions.

has been done in order to maintain the notation of related work (see e.g. Refs. [1,11] for an explanation). On the other hand, the term signal will always refer to a scalar quantity (e.g. the acceleration in one direction at a given point as mentioned above).

The MISO-network corresponds to a special case of the N-dimensional linear network (see Fig. 1). The signal at one particular subsystem with no force acting on it is chosen as the *output* while the signals or forces at the remaining ones are termed the problem *inputs*. Signals will be used as inputs in the expressions derived below, although these expressions remain valid for forces. Note that if $N-1$ forces are entering the system, $N-1$ signals have to be used if the method is not to be applied using forces (see Fig. 1). The output is identified with the Nth subsystem of the network and it will be often symbolized with the subscript y following the notation in Ref. [1]. The method aims at finding the frequency response functions between every input and the output (see Fig. 1). The output can be linked to the inputs by means of the linear relation[1] (see Ref. [1])

$$\hat{s}_y = \sum_{i \neq y}^{N-1} \hat{\mathbf{H}}_{iy} \hat{s}_i, \tag{1}$$

where $\hat{\mathbf{H}}_{iy}$ are the unknown frequency response functions, $\hat{s}_i$ is the signal at the ith network subsystem, $\hat{s}_y$ denotes the signal at the output and the circumflex superscript denotes frequency dependence. In order to find the unknowns $\hat{\mathbf{H}}_{iy}$, both sides of Eq. (1) are multiplied by $\hat{s}_j^*$ and then averaged (the ensemble average or the time average under the ergodic assumption can be considered, see e.g. Refs. [1,12] for details) to arrive at

$$\bar{\hat{\mathbf{S}}}_{jy} = \sum_{i \neq y}^{N-1} \hat{\mathbf{H}}_{iy} \bar{\hat{\mathbf{S}}}_{ji}, \quad j \neq y. \tag{2}$$

Similarly, it is possible to multiply both sides of Eq. (1) by $\hat{s}_j^*$ and then average to obtain the auto-spectrum at the output

$$\bar{\hat{\mathbf{S}}}_{yy} = \sum_{i \neq y}^{N-1} \hat{\mathbf{H}}_{iy} \bar{\hat{\mathbf{S}}}_{yi}. \tag{3}$$

[1] In Ref. [1] a noise term is added to Eq. (1), which has not been included here for the sake of simplicity.

In Eq. (2), $\bar{\hat{\mathbf{S}}}_{ji}$ stands for

$$\bar{\hat{\mathbf{S}}}_{ji} = \overline{\hat{s}_j^* \hat{s}_i} = \frac{1}{M} \sum_{k=1}^{M} \hat{s}_{jk}^* \hat{s}_{ik} \tag{4}$$

and analogous expressions are assumed for $\bar{\hat{\mathbf{S}}}_{jy}$, and for $\bar{\hat{\mathbf{S}}}_{yy}$ and $\bar{\hat{\mathbf{S}}}_{yi}$ in Eq. (3). M is the number of samples. Eq. (2) can be solved to find $\hat{\mathbf{H}}_{iy}$ once the various signal cross-spectra have been measured. Then, Eqs. (1) and (3) can be used to find the signal and the auto-spectrum factorizations, respectively, at the output. The range of validity of this procedure can be better understood if Eq. (2) is written in matrix form

$$\begin{pmatrix} \bar{\hat{\mathbf{S}}}_{1y} \\ \bar{\hat{\mathbf{S}}}_{2y} \\ \vdots \\ \bar{\hat{\mathbf{S}}}_{N-1y} \end{pmatrix} = \begin{pmatrix} \bar{\hat{\mathbf{S}}}_{11} & \bar{\hat{\mathbf{S}}}_{12} & \cdots & \bar{\hat{\mathbf{S}}}_{1N-1} \\ \bar{\hat{\mathbf{S}}}_{21} & \bar{\hat{\mathbf{S}}}_{22} & \cdots & \bar{\hat{\mathbf{S}}}_{2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\hat{\mathbf{S}}}_{N-11} & \bar{\hat{\mathbf{S}}}_{N-12} & \cdots & \bar{\hat{\mathbf{S}}}_{N-1N-1} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{H}}_{1y} \\ \hat{\mathbf{H}}_{2y} \\ \vdots \\ \hat{\mathbf{H}}_{N-1y} \end{pmatrix}. \tag{5}$$

When trying to solve Eq. (5) three different cases occur (see e.g. Ref. [13]). First, the problem cannot be solved if the subsystem signals are fully correlated because the matrix columns become linearly dependent and consequently, the matrix determinant turns out to be null. Second, if the subsystem signals are fully uncorrelated, the matrix system is easily solved as $\bar{\hat{\mathbf{S}}}$ becomes diagonal. This yields

$$\hat{\mathbf{H}}_{iy} = \frac{\bar{\hat{\mathbf{S}}}_{iy}}{\bar{\hat{\mathbf{S}}}_{ii}} \quad \forall i = 1, \ldots, N-1. \tag{6}$$

Third, if the subsystem signals are partially correlated, the matrix $\bar{\hat{\mathbf{S}}}$ is no longer diagonal and the linear system (5) has to be solved to obtain the various $\hat{\mathbf{H}}_{iy}$. Several algorithms exist to invert $\bar{\hat{\mathbf{S}}}$ and solve Eq. (5). The conditioned spectral density functions developed in Refs. [1,14,15] can be obtained in fact, as a by-pass product of a particular procedure of solving the linear system (5). As mentioned above, it is the purpose of this paper to show that all the terms arising from the conditioned spectral analysis can be obtained in a compact form by means of the $\mathbf{LDL}^H$-factorization of the cross-spectral density matrix $\bar{\hat{\mathbf{S}}}$. The $\mathbf{LDL}^H$-factorization of a matrix can be obtained straightforwardly from its $\mathbf{LU}$-factorization (see Ref. [16]) that is one possible way to solve Eq. (5). The importance of the conditioned spectral density functions relies on the fact that they give information on the noise and vibration transmission paths among the network subsystems. That is, in some cases for instance, they can determine the amount of the signal at the output that is due to the direct link between the output and say, subsystem 1, and how much of the output signal comes from the remaining subsystems.

2. Conditioned spectral density functions

2.1. Compact formulation by means of the $\mathbf{LDL}^H$-factorization

The signal at subsystem i when all linear effects from the remaining subsystem signals $\hat{s}_1 \ldots \hat{s}_{i-1}$ have been removed from it by using least-squares techniques (see Refs. [1,12]) will be denoted by $\hat{s}_{i\bullet 1\ldots i-1}$. In an analogous way, $\hat{S}_{ij\bullet 1}$ will denote the conditioned cross-spectral density function between subsystems i and j without passing through subsystem 1. $\hat{S}_{ij\bullet 1}$ can be obtained from

$$\hat{S}_{ij\bullet 1} := \frac{\bar{\hat{\mathbf{S}}}_{11}\bar{\hat{\mathbf{S}}}_{ij} - \bar{\hat{\mathbf{S}}}_{i1}\bar{\hat{\mathbf{S}}}_{1j}}{\bar{\hat{\mathbf{S}}}_{11}}. \tag{7}$$

That is to say, Eq. (7) represents the cross-spectral density function between i and j when the linear effects of $\hat{s}_1$ are removed from $\hat{s}_i$ and $\hat{s}_j$. This equation can be generalized so that the conditioned cross-spectral density function between any pair of subsystems i and j, when the linear effects of the subsystems set $\{1\ldots n\}$ has been removed from their signals, turns out to be

$$\hat{S}_{ij\bullet 1\ldots n} = \frac{\hat{S}_{nn\bullet 1\ldots n-1}\hat{S}_{ij\bullet 1\ldots n-1} - \hat{S}_{in\bullet 1\ldots n-1}\hat{S}_{nj\bullet 1\ldots n-1}}{\hat{S}_{nn\bullet 1\ldots n-1}},$$
$$i = n+2, \ldots, N, \;\; j = n+2, \ldots, N. \tag{8}$$

By defining the parameter $\hat{\mathbf{L}}_{ij}$ as

$$\hat{\mathbf{L}}_{ij} := \frac{\hat{S}_{ij\bullet 1\ldots i-1}}{\hat{S}_{ii\bullet 1\ldots i-1}}, \quad i,j = 1, \ldots, N \tag{9}$$

it follows that (see Ref. [14])

$$\bar{\hat{\mathbf{S}}}_{yy} = \sum_{i=1}^{N-1} \left|\mathbf{L}_{iy}\right|^2 \hat{S}_{ii\bullet 1\ldots i-1} + \hat{S}_{yy\bullet 1\ldots N-1} \tag{10}$$

and the signal at the receiver, $\hat{s}_y$, can be obtained from (see Fig. 2)

$$\hat{s}_y = \sum_{i=1}^{N} \hat{\mathbf{L}}_{iy}\hat{s}_{i\bullet 1\ldots i-1}. \tag{11}$$

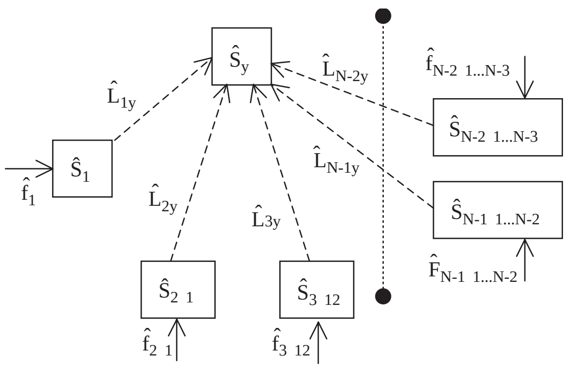

Fig. 2. The MISO network. Second factorization: signal or force conditioned spectral density functions.

Eq. (11) gives the signal at y as the summation of various contributions: contribution from subsystem 1, plus contribution from subsystem 2 when the linear effects from 1 to 2 have been removed, plus contribution of subsystem 3 when the linear effects from 1 and 2 to 3 have been removed, etc. Hence, Eqs. (1) and (11) allow the signal at the output from the original and conditioned signals at all the remaining subsystems, respectively, to be reconstructed once the parameters $\hat{\mathbf{H}}_{iy}$ and $\hat{\mathbf{L}}_{iy}$ have been found. The same follows for the auto-spectra in Eqs. (3) and (10).

The relations between $\hat{S}_{ij\bullet 1\div n}$ and $\hat{\mathbf{H}}_{ij}$ and between $\hat{\mathbf{H}}_{iy}$ and $\hat{\mathbf{L}}_{iy}$ are given in Refs. [14,15] with some minor differences concerning the notation used here and the addition of the noise term mentioned previously. These relations are given by

$$\hat{\mathbf{H}}_{ij} = \frac{\hat{S}_{ij\bullet 1\ldots N_{ij}}}{\hat{S}_{ii\bullet 1\ldots N_{ij}}}, \quad N_{ij} := \{N\} - \{i,j\}, \tag{12}$$

$$\hat{\mathbf{L}}_{iy} = \sum_{j}^{N-1} \hat{\mathbf{L}}_{ij}\hat{\mathbf{H}}_{iy}, \quad \hat{\mathbf{L}}_{N-1y} = \hat{\mathbf{H}}_{N-1y}. \tag{13}$$

As mentioned above, the conditioned spectral density functions approach and their corresponding equations may be easily understood as naturally arising from the process involved in the solution of the linear system (5). If an **LU**-factorization of the cross-spectra density matrix $\bar{\hat{\mathbf{S}}}$ is performed to do so, the parameters $\hat{\mathbf{L}}_{ij}$ and the conditioned signals are recovered. By including an extra row and column in $\hat{\mathbf{S}}$, corresponding to the output (Nth subsystem), it follows from the $\mathbf{LDL}^H$-factorization of $\bar{\hat{\mathbf{S}}}$ that

$$\bar{\hat{\mathbf{S}}} = \mathbf{LDL}^H$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{\hat{S}_{21}}{\hat{S}_{11}} & 1 & 0 & 0 & \cdots & 0 \\ \frac{\hat{S}_{31}}{\hat{S}_{11}} & \frac{\hat{S}_{32\bullet 1}}{\hat{S}_{22\bullet 1}} & 1 & 0 & \cdots & 0 \\ \frac{\hat{S}_{41}}{\hat{S}_{11}} & \frac{\hat{S}_{42\bullet 1}}{\hat{S}_{22\bullet 1}} & \frac{\hat{S}_{43\bullet 12}}{\hat{S}_{33\bullet 12}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{\hat{S}_{N1}}{\hat{S}_{11}} & \frac{\hat{S}_{N2\bullet 1}}{\hat{S}_{22\bullet 1}} & \frac{\hat{S}_{N3\bullet 12}}{\hat{S}_{33\bullet 12}} & \cdots & \frac{\hat{S}_{NN-1\bullet 1\ldots N-2}}{\hat{S}_{N-1N-1\bullet 1\ldots N-2}} & 1 \end{bmatrix} \begin{bmatrix} \hat{S}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{S}_{22\bullet 1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \hat{S}_{33\bullet 12} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \hat{S}_{44\bullet 123} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{S}_{NN\bullet 1\ldots N-1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \frac{\hat{S}_{12}}{\hat{S}_{11}} & \frac{\hat{S}_{13}}{\hat{S}_{11}} & \frac{\hat{S}_{14}}{\hat{S}_{11}} & \cdots & \frac{\hat{S}_{1N}}{\hat{S}_{11}} \\ 0 & 1 & \frac{\hat{S}_{23\bullet 1}}{\hat{S}_{22\bullet 1}} & \frac{\hat{S}_{24\bullet 1}}{\hat{S}_{22\bullet 1}} & \cdots & \frac{\hat{S}_{2N\bullet 1}}{\hat{S}_{22\bullet 1}} \\ 0 & 0 & 1 & \frac{\hat{S}_{34\bullet 12}}{\hat{S}_{33\bullet 12}} & \cdots & \frac{\hat{S}_{3N\bullet 12}}{\hat{S}_{33\bullet 12}} \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \frac{\hat{S}_{N-1N\bullet 1\ldots N-2}}{\hat{S}_{N-1N-1\bullet 1\ldots N-2}} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \hat{\mathbf{L}}_{12}^* & 1 & 0 & 0 & \cdots & 0 \\ \hat{\mathbf{L}}_{13}^* & \hat{\mathbf{L}}_{23}^* & 1 & 0 & \cdots & 0 \\ \hat{\mathbf{L}}_{14}^* & \hat{\mathbf{L}}_{24}^* & \hat{\mathbf{L}}_{34}^* & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \hat{\mathbf{L}}_{1N}^* & \hat{\mathbf{L}}_{2N}^* & \hat{\mathbf{L}}_{3N}^* & \cdots & \hat{\mathbf{L}}_{N-1N}^* & 1 \end{bmatrix} \begin{bmatrix} \hat{S}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{S}_{22\bullet 1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \hat{S}_{33\bullet 12} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \hat{S}_{44\bullet 123} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{S}_{NN\bullet 1\ldots N-1} \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & \hat{\mathbf{L}}_{12} & \hat{\mathbf{L}}_{13} & \hat{\mathbf{L}}_{14} & \cdots & \hat{\mathbf{L}}_{1N} \\ 0 & 1 & \hat{\mathbf{L}}_{23} & \hat{\mathbf{L}}_{24} & \cdots & \hat{\mathbf{L}}_{2N} \\ 0 & 0 & 1 & \hat{\mathbf{L}}_{34} & \cdots & \hat{\mathbf{L}}_{3N} \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \hat{\mathbf{L}}_{N-1N} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{14}
$$

where use of $\hat{S}^*_{ij\bullet 1\ldots n} = \hat{S}_{ji\bullet 1\ldots n}$ and $\hat{\mathbf{L}}^*_{ij} := \hat{S}^*_{ij\bullet 1\ldots i-1}/\hat{S}^*_{ii\bullet 1\ldots i-1}$ has been made in the last equality, and the over bar denoting mean values on matrix components has been omitted to simplify the notation. Note that in order to keep the original notations in Refs. [1,14,16] a certain inconsistency has arisen, since elements of matrix $\mathbf{L}$ in Eq. (14) are given by $\hat{\mathbf{L}}^*_{ji}$ instead of $\hat{\mathbf{L}}_{ij}$. Nevertheless, this has no further complication and does not influence the foregoing results. Eq. (14) will be proven in next Section.

From Eq. (14), $\hat{S}$ can be expanded as

$$
\bar{\hat{\mathbf{S}}} = \sum_{i=1}^{N} \hat{S}_{ii\bullet 1\ldots i-1}\mathbf{l}_i(\mathbf{l}_i)^H = \hat{S}_{11}\mathbf{l}_1(\mathbf{l}_1)^H + \hat{S}_{22\bullet 1}\mathbf{l}_2(\mathbf{l}_2)^H + \hat{S}_{33\bullet 12}\mathbf{l}_3(\mathbf{l}_3)^H + \cdots + \hat{S}_{NN\bullet 1\ldots N-1}\mathbf{l}_N(\mathbf{l}_N)^H, \tag{15}
$$

where $\mathbf{l}_i$ denotes the ith column vector of matrix $\mathbf{L}$ and $\mathbf{l}_i^H$ its adjoint. The series development for the various elements in Eq. (15) allow e.g. Eq. (10) to be recovered and give rise to the various transitional relations of the method (see Refs. [14,15]). Note also that the identity $tr(\bar{\hat{\mathbf{S}}}) = tr(\mathbf{LDL}^H)$ (with $tr(\cdot)$ denoting the matrix trace) states the obvious fact that the whole energy of the system does not depend on how it is factorized among the various subsystems.

2.2. Proof

The proof of Eq. (14) in the previous Section starts by finding the **LU**-factorization of matrix $\bar{\hat{\mathbf{S}}}$, i.e. factorizing $\bar{\hat{\mathbf{S}}}$ as a product of a unit lower triangular matrix, $\mathbf{L}$, (with units on the main diagonal) and a regular upper triangular matrix, $\mathbf{U}$.

The first result will show that the $N \times N$ matrix $\mathbf{U}$ is given by the following expression:

$$\mathbf{U} = \begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} & \hat{S}_{13} & \hat{S}_{14} & \cdots & \hat{S}_{1N} \\ 0 & \hat{S}_{22\bullet 1} & \hat{S}_{23\bullet 1} & \hat{S}_{24\bullet 1} & \cdots & \hat{S}_{2N\bullet 1} \\ 0 & 0 & \hat{S}_{33\bullet 12} & \hat{S}_{34\bullet 12} & \cdots & \hat{S}_{3N\bullet 12} \\ 0 & 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \hat{S}_{NN\bullet 1\ldots N-1} \end{bmatrix}. \tag{16}$$

Note that again, over bars denoting averages on matrix elements have not been explicitly written. It is shown in Ref. [16] that the $N \times N$ upper triangular matrix $\mathbf{U}$ of $\bar{\hat{\mathbf{S}}}$ can be written in terms of a product series of the so-called gaussian matrices. That is

$$\mathbf{U} = \mathbf{M}_{N-1}\mathbf{M}_{N-2}\ldots\mathbf{M}_2\mathbf{M}_1\bar{\hat{\mathbf{S}}} = \left(\prod_{i=1}^{N-1} \mathbf{M}_{N-i}\right)\bar{\hat{\mathbf{S}}}, \tag{17}$$

where $\mathbf{M}_i$ is given by

$$\mathbf{M}_i = \mathbf{I} - \mathbf{t}^i\mathbf{e}_i^{\mathrm{T}}, \tag{18}$$

$\mathbf{I}$ is the identity matrix, $\mathbf{t}^i = (0 \ldots_{i\ zeros}\ 0, \tau_{i+1} \ldots \tau_N)^{\mathrm{T}}$ is the ith *Gauss vector* whose components $\tau(i+1:n)$ are the so-called *Gauss multipliers*, and $\mathbf{e}_i$ is the ith vector of the canonical base.

If the matrix $\mathbf{U}^n$ is identified with the product of the first n terms in Eq. (18),

$$\mathbf{U}^n \equiv \mathbf{M}_n\mathbf{M}_{n-1}\ldots\mathbf{M}_2\mathbf{M}_1\bar{\hat{\mathbf{S}}}, \quad 1 < n \leqslant N-1 \tag{19}$$

the multipliers of the matrix $\mathbf{M}_{n+1}$ can be obtained from the components of the $\mathbf{U}^n$ matrix:

$$\mathbf{t}^{n+1} = \left(0 \underset{n+1\ \text{zeros}}{\ldots} 0, \frac{u^n_{n+2n+1}}{u^n_{n+1n+1}} \cdots \frac{u^n_{Nn+1}}{u^n_{n+1n+1}}\right)^{\mathrm{T}}. \tag{20}$$

Particular cases of Eq. (19) are $\mathbf{U}^{N-1} = \mathbf{U}$ and $\mathbf{U}^1 = \mathbf{M}_1\bar{\hat{\mathbf{S}}}$. On the other hand, note that Eq. (20) requires that $U^n_{n+1n+1} \neq 0$ for any n.

It will be proven by induction that $\mathbf{U}^n$ has the following expression:

$$\mathbf{U}^n = \begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} & \hat{S}_{13} & \cdots & \hat{S}_{1n} & \hat{S}_{1n+1} & \cdots & \hat{S}_{1N} \\ 0 & \hat{S}_{22\bullet 1} & \hat{S}_{23\bullet 1} & \cdots & \hat{S}_{2n\bullet 1} & \hat{S}_{2n+1\bullet 1} & \cdots & \hat{S}_{2N\bullet 1} \\ \vdots & 0 & \hat{S}_{33\bullet 12} & \cdots & \hat{S}_{3n\bullet 12} & \hat{S}_{3n+1\bullet 12} & \cdots & \hat{S}_{3N\bullet 12} \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \hat{S}_{nn\bullet 1\ldots n-1} & \hat{S}_{nn+1\bullet 1\ldots n-1} & \cdots & \hat{S}_{nN\bullet 1\ldots n-1} \\ \vdots & \vdots & \vdots & \vdots & 0 & \hat{S}_{n+1n+1\bullet 1\ldots n} & \cdots & \hat{S}_{nN\bullet 1\ldots n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \hat{S}_{Nn+1\bullet 1\ldots n} & \cdots & \hat{S}_{NN\bullet 1\ldots n} \end{bmatrix}. \tag{21}$$

Case $n = 1$: It will be first shown that Eq. (21) holds for $n = 1$. From Eq. (19) it follows $\mathbf{U}^1 = \mathbf{M}_1\hat{\mathbf{S}}$, and Eqs. (18) and (20) yield

$$\mathbf{M}_1 = \mathbf{I} - \begin{pmatrix} 0 \\ \hat{S}_{21}/\hat{S}_{11} \\ \hat{S}_{31}/\hat{S}_{11} \\ \vdots \\ \hat{S}_{N1}/\hat{S}_{11} \end{pmatrix} (0\ 1\ 0\ \cdots\ 0) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\hat{S}_{21}/\hat{S}_{11} & 1 & 0 & 0 & \cdots & 0 \\ -\hat{S}_{31}/\hat{S}_{11} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ -\hat{S}_{N1}/\hat{S}_{11} & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{22}$$

Performing the matrix product of $\mathbf{M}_1$ and $\hat{\mathbf{S}}$ for the elements of matrix $\mathbf{U}^1$ results in

$$\begin{aligned} U^1_{ij} &= \hat{S}_{ij}, \quad i = 1,\ j = 1, \ldots, N \\ U^1_{ij} &= 0, \quad i = 1, \ldots, N,\ j = 1 \\ U^1_{ij} &= \hat{S}_{ij} - \frac{\hat{S}_{i1}\hat{S}_{1j}}{\hat{S}_{11}}, \quad i = 2, \ldots, N,\ j = 2, \ldots, N \end{aligned} \tag{23}$$

and from the recursive relation for the conditional cross-spectra functions (8) it can be seen that the last equality of Eq. (23) becomes

$$U^1_{ij} = \hat{S}_{ij} - \frac{\hat{S}_{i1}\hat{S}_{1j}}{\hat{S}_{11}} = \frac{\hat{S}_{ij}\hat{S}_{11} - \hat{S}_{i1}\hat{S}_{1j}}{\hat{S}_{11}} = \hat{S}_{ij\bullet 1}, \quad i = 2, \ldots, N,\ j = 2, \ldots, N \tag{24}$$

so it is now clear that $\mathbf{U}^1$ can be obtained from Eq. (21) with $n = 1$.

Case $n + 1$: It will be now proven that if $\mathbf{U}^n$ is given by Eq. (21) then $\mathbf{U}^{n+1}$ can be obtained from the same expression. From Eq. (19) it follows that

$$\mathbf{U}^{n+1} = \mathbf{M}_{n+1}\mathbf{U}_n \tag{25}$$

and using Eqs. (18) and (20) $\mathbf{M}_{n+1}$ becomes

$$\mathbf{M}_{n+1} = \mathbf{I} - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\hat{S}_{n+2n+1\bullet 1\ldots n-1}}{\hat{S}_{n+1n+1\bullet 1\ldots n-1}} \\ \vdots \\ \frac{\hat{S}_{Nn+1\bullet 1\ldots n-1}}{\hat{S}_{n+1n+1\bullet 1\ldots n-1}} \end{pmatrix} (0\ \cdots\ 0\ 1\ 0\ \cdots\ 0)$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 1 & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\frac{\hat{S}_{n+2n+1\bullet 1\ldots n-1}}{\hat{S}_{n+1n+1\bullet 1\ldots n-1}} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 0 \\ 0 & \cdots & 0 & -\frac{\hat{S}_{Nn+1\bullet 1\ldots n-1}}{\hat{S}_{n+1n+1\bullet 1\ldots n-1}} & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{26}$$

Performing the matrix product (25) for the $\mathbf{U}^{n+1}$ elements yields

$$\begin{aligned} &U_{ij}^{n+1} = U_{ij}^{n}, \quad i = 1, \ldots, n+1, \ j = 1, \ldots, n+1, \\ &U_{ij}^{n+1} = 0, \quad i = n+2, \ldots, N, \ j = 1, \ldots, n+1, \\ &U_{ij}^{n+1} = \hat{S}_{ij\bullet 1\ldots n-1} - \frac{\hat{S}_{in\bullet 1\ldots n-1}\hat{S}_{nj\bullet 1\ldots n-1}}{\hat{S}_{nn\bullet 1\ldots n-1}}, \quad i = n+2, \ldots, N, \ j = n+2, \ldots, N \end{aligned} \tag{27}$$

and using again the recursive relation for the conditional cross-density functions (8), the last equality of Eq. (27) can be written as

$$\begin{aligned} U_{ij}^{n+1} = \hat{S}_{ij\bullet 1\ldots n-1} - \frac{\hat{S}_{in\bullet 1\ldots n-1}\hat{S}_{nj\bullet 1\ldots n-1}}{\hat{S}_{nn\bullet 1\ldots n-1}} = \hat{S}_{ij\bullet 1\ldots n}, \\ i = n+2, \ldots, N, \ j = n+2, \ldots, N. \end{aligned} \tag{28}$$

So it has finally been shown that Eq. (21) holds for $\mathbf{U}^{n+1}$ if it holds for $\mathbf{U}^{n}$, which proves its validity together with the result for the case $n = 1$.

Once $\mathbf{U}$ is obtained, it is necessary to calculate the unity lower matrix $\mathbf{L}$. $\mathbf{L}$ is given by the following expression:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{\hat{S}_{21}}{\hat{S}_{11}} & 1 & 0 & 0 & \cdots & 0 \\ \frac{\hat{S}_{31}}{\hat{S}_{11}} & \frac{\hat{S}_{32\bullet 1}}{\hat{S}_{22\bullet 1}} & 1 & 0 & \cdots & 0 \\ \frac{\hat{S}_{41}}{\hat{S}_{11}} & \frac{\hat{S}_{42\bullet 1}}{\hat{S}_{22\bullet 1}} & \frac{\hat{S}_{43\bullet 12}}{\hat{S}_{33\bullet 12}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{\hat{S}_{N1}}{\hat{S}_{11}} & \frac{\hat{S}_{N2\bullet 1}}{\hat{S}_{22\bullet 1}} & \frac{\hat{S}_{N3\bullet 12}}{\hat{S}_{33\bullet 12}} & \cdots & \frac{\hat{S}_{NN-1\bullet 1\ldots N-2}}{\hat{S}_{N-1N-1\bullet 1\ldots N-2}} & 1 \end{bmatrix}. \tag{29}$$

A series product development of $\mathbf{L}$ can be used to prove (29) (see [16]). $\mathbf{L}$ can be obtained from the inverses of the various gaussian matrices appearing in Eq. (17)

$$\mathbf{L} = \mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\ldots\mathbf{M}_{N-2}^{-1}\mathbf{M}_{N-1}^{-1} = \prod_{i=1}^{N-1} \mathbf{M}_i^{-1}. \tag{30}$$

These inverses can be easily calculated taking into account that

$$\mathbf{M}_i^{-1} = \mathbf{I} + \mathbf{t}^i\mathbf{e}_i^{\mathrm{T}}. \tag{31}$$

In order to finally prove that $\mathbf{L}$ has the expression in Eq. (29), an induction process very similar to the one followed to obtain $\mathbf{U}$ can be applied to Eq. (30). As no new insight is gained, this process is not performed here.

From the above deductions the $\mathbf{LU}$-factorization of $\bar{\hat{\mathbf{S}}}$ has been obtained, hence

$$\bar{\hat{\mathbf{S}}} = \mathbf{LU} \tag{32}$$

with $\mathbf{L}$ given by Eq. (29) and $\mathbf{U}$ given by Eq. (21). Obtaining the $\mathbf{LDL}^H$-factorization of $\bar{\hat{\mathbf{S}}}$ is now straightforward. $\mathbf{U}$ is written as a product of two matrices: a diagonal matrix $\mathbf{D}$ whose terms are given by the diagonal of $\mathbf{U}$ and another matrix, $\mathbf{U}'$, which is unity upper triangular and whose rows are those of $\mathbf{U}$ divided by the corresponding row-element of $\mathbf{D}$. It follows

$$\mathbf{U} = \mathbf{DU}' \tag{33}$$

and from Eqs. (16), (29) and (4)

$$\mathbf{U}' = \mathbf{L}^H. \tag{34}$$

Hence the $\mathbf{LDL}^H$-factorization of $\bar{\hat{\mathbf{S}}}$ Eq. (14) has been obtained.

3. Conclusions

In this paper, it has been shown that the conditioned spectral density function analysis developed to deal with partially correlated signals on a linear network can be obtained from the $\mathbf{LDL}^H$ factorization of the network signal cross-spectra matrix $\hat{\mathbf{S}}$. This allows all the terms appearing in the series development of the conditioned spectral analysis to be obtained in a clear and compact formulation. As most modern mathematical software packages contain matrix factorizations such as the $\mathbf{LU}$ or the $\mathbf{LDL}^H$ ones, it is quite a straightforward matter to carry out a conditioned spectral analysis from measured data.

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